Problem 2.16

A golfer hits his ball with speed v_o at an angle θ above the horizontal ground. Assuming that the angle θ is fixed and that air resistance can be neglected, what is the minimum speed $v_o(\text{min})$ for which the ball will clear a wall of height h, a distance d away? Your solution should get into trouble if the angle θ is such that $\tan\theta < h/d$. Explain. What is $v_o(\text{min})$ if $\theta = 25°$, $d = 50$ m, and $h = 2$ m?

[TYPO: The solution actually gets into trouble when $\tan\theta \leq h/d$.]

Solution

Newton's second law gives two equations of motion, one for each dimension the projectile moves in.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_x = ma_x \\ \sum F_y = ma_y \end{cases}$$

Since there's no air resistance, the only force to consider is the one due to gravity.

$$\begin{cases} 0 = ma_x \\ -mg = ma_y \end{cases}$$

$$\begin{cases} a_x = 0 \\ a_y = -g \end{cases}$$

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\ \dfrac{d^2y}{dt^2} = -g \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = v_o\cos\theta \\ \dfrac{dy}{dt} = -gt + v_o\sin\theta \end{cases}$$

$$\begin{cases} x(t) = (v_o\cos\theta)t + x_o \\ y(t) = -\dfrac{1}{2}gt^2 + (v_o\sin\theta)t + y_o \end{cases}$$

Take the launching site to be the origin so that $x_o = 0$ and $y_o = 0$.

$$\begin{cases} x(t) = (v_o \cos\theta)t \\ \\ y(t) = -\frac{1}{2}gt^2 + (v_o \sin\theta)t \end{cases}$$

The projectile needs to cover a horizontal distance of d and a vertical height of h to clear the wall.

$$\begin{cases} d = (v_o \cos\theta)t \\ \\ h = -\frac{1}{2}gt^2 + (v_o \sin\theta)t \end{cases}$$

This is a system of two equations for two unknowns (t and v_o). Solve the first equation for t,

$$t = \frac{d}{v_o \cos\theta},$$

and plug it into the second equation to get v_o.

$$h = -\frac{1}{2}g\left(\frac{d}{v_o \cos\theta}\right)^2 + (v_o \sin\theta)\left(\frac{d}{v_o \cos\theta}\right)$$

$$h = -\frac{gd^2}{2v_o^2 \cos^2\theta} + d\tan\theta$$

$$\frac{gd^2}{2v_o^2 \cos^2\theta} = d\tan\theta - h$$

$$\frac{1}{v_o^2} = \frac{2\cos^2\theta}{gd^2}(d\tan\theta - h)$$

$$v_o^2 = \frac{gd^2}{2\cos^2\theta(d\tan\theta - h)}$$

$$v_o(\text{min}) = \sqrt{\frac{gd^2}{2\cos^2\theta(d\tan\theta - h)}}$$

Notice that in order for this solution to be valid and physically reasonable, it's necessary that

$$d\tan\theta - h > 0$$

$$\tan\theta > \frac{h}{d}.$$

If $d\tan\theta - h < 0$, then there will be a negative number under the square root, resulting in an imaginary velocity. If $d\tan\theta - h = 0$, then the denominator is zero, blowing up the velocity.

Either of these conditions being true means that the ball will not clear the wall. If $\theta = 25°$, $d = 50$ m, and $h = 2$ m, then

$$v_{\text{o}}(\text{min}) = \sqrt{\frac{\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)(50\ \text{m})^2}{2(\cos 25°)^2[(50\ \text{m})\tan 25° - (2\ \text{m})]}} \approx 26.5\ \frac{\text{m}}{\text{s}}.$$